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The quantum algebra $U_q(\mathrm{su}_2)$ and q-Krawtchouk families of polynomials

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Abstract
The aim of this paper is to study quantum and affine q-Krawtchouk polynomials by means of operators of irreducible representations of the quantum algebra $U_q(\mathrm{su}_2)$. We diagonalize a certain operator I in such a representation and show that elements of the transition matrix from the initial (canonical) basis to the basis consisting of eigenfunctions of the operator I are expressed in terms of quantum q-Krawtchouk polynomials. Then we find an explicit form of the operator q^{J_3} in the basis of the eigenfunctions of I, in which it has the form of a Jacobi matrix. Normalizing this basis and using the operator q^{J_3}, we thus derive the orthogonality relations for quantum and affine q-Krawtchouk polynomials. We show that affine q-Krawtchouk polynomials are dual to quantum q-Krawtchouk polynomials. A biorthogonal system of functions (with respect to the scalar product in the representation space) is also derived.

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1. Introduction

In 1929, Krawtchouk [1] introduced polynomials orthogonal on a finite set. It was shown by Koornwinder [2] that these polynomials are closely connected with irreducible unitary representations of the Lie group $SU(2)$. They appear when we consider matrix elements of these representations as functions of a number of a row of the representation matrix in the canonical (standard) basis.

The different types of q-Krawtchouk polynomials are related to representations of the quantum group $SU_q(2)$ and the quantum algebra $U_q(\mathrm{su}_2)$ (see [3–6]). It is known that for integral values of q, q-Krawtchouk polynomials are connected with representations of Chevalley groups (see, for example, [7]). It was shown recently how q-Krawtchouk polynomials are related to spherical functions on the Hecke algebra of type b (see [8]).

The aim of this paper is to show how quantum and affine q-Krawtchouk polynomials are connected in a simple way with the representation theory of the quantum algebra $U_q(\mathrm{su}_2)$. (Note that the theory of representations of this algebra is simpler than the corresponding theory for the quantum group $SU_q(2)$, and this makes more attractive to study quantum q-Krawtchouk polynomials by means of representations of the algebra $U_q(\mathrm{su}_2)$. In that case we do not need the Hopf algebra structure of this algebra.) We use the method of the papers [9, 10], where by means of representations of the algebra $U_q(\mathrm{su}_{1,1})$ other types of q-orthogonal polynomials have been investigated.

In this paper we show how one can prove in a simple way (by using finite-dimensional irreducible representations of the quantum algebra $U_q(\mathrm{su}_2)$) the orthogonality relations for quantum and affine q-Krawtchouk polynomials. To achieve this we need two operators in a representation (one is diagonal in the canonical basis and the second, denoted by I, has the form of a Jacobi matrix in this basis; note that these two operators constitute a Leonard pair, see [11] for a definition). The quantum q-Krawtchouk polynomials appear as entries of the transition matrix from the canonical basis to the basis consisting of eigenfunctions of the operator I. We normalize the last basis and obtain this transition matrix as a unitary matrix. Then orthogonalities of rows and columns in this matrix give orthogonality relations for quantum and affine q-Krawtchouk polynomials, respectively. In this way, affine q-Krawtchouk polynomials are duals to quantum q-Krawtchouk polynomials (although affine q-Krawtchouk polynomials are self-dual).

Note that there exists another motivation for studying the operator I in an irreducible representation of $U_q(\mathrm{su}_2)$, important for applications in physics. Many models of quantum optics, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single-mode radiation field (Dicke model), can be described by interaction Hamiltonians of the form I (see [12] and references therein). From this point of view it is necessary to have a detailed knowledge about operators of such type (diagonalization, eigenvalues, eigenfunctions, etc). Therefore, we perform a detailed study of the operator I in this paper.

Throughout the sequel we assume that q is a fixed number such that $0 < q < 1$. We use the theory of special functions and notations of the standard q-analysis (see, for example, [13]). Our definition of q-numbers $[a]_q$ is as follows

$$[a]_q = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}$$

where a is any complex number or an operator.

2. The quantum algebra $U_q(\mathrm{su}_2)$ and its representations

The quantum algebra $U_q(\mathrm{su}_2)$ is an associative algebra, generated by the elements J_+, J_- and J_3, satisfying the relations

$$[J_+, J_-] = [2J_3]_q \qquad [J_3, J_\pm] = \pm J_\pm.$$

Nontrivial finite-dimensional irreducible representations of the algebra $U_q(\mathrm{su}_2)$ are given by positive integers or half-integers j (see [14], chapter 3). We denote such a representation, acting in $(2j + 1)$-dimensional space, by T_j.

The linear space of the irreducible representation T_j can be realized as the space $\mathcal{H}_j$ of all polynomials in x of degree less than or equal to $2j$. The operators J_3 and $J_\pm$ are realized in this space as

$$J_3 = x\frac{\mathrm{d}}{\mathrm{d}x} - j \qquad J_+ = x\left[2j - x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q \qquad J_- = \frac{1}{x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q$$

(see [5] and [6]). The canonical basis of the space $\mathcal{H}_j$ consists of monomials

$$f_m^j(x) = c_m^j x^{j+m} \qquad m = -j, -j+1, \ldots, j \qquad c_m^j = q^{(m^2-j^2)/4} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} \tag{1}$$

where the q-binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is defined as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q,q)_m}{(q,q)_n (q,q)_{m-n}} = (-1)^n q^{mn-n(n-1)/2} \frac{(q^{-m};q)_n}{(q;q)_n}$$

and $(a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$.

We introduce a scalar product $\langle \cdot, \cdot \rangle$ into $\mathcal{H}_j$, assuming that $\left\langle f_m^j, f_n^j \right\rangle = \delta_{mn}$. This turns $\mathcal{H}_j$ into a finite-dimensional Hilbert space. In the canonical basis (1) the operators J_3 and $J_\pm$ act as

$$J_+ f_m^j = \sqrt{[j+m+1]_q [j-m]_q}\, f_{m+1}^j = \frac{q^{(j-n+1/2)/2}}{1-q} \sqrt{(1-q^{n+1})(q^{n-2j}-1)}\, f_{m+1}^j$$

$$J_- f_m^j = \sqrt{[j-m+1]_q [j+m]_q}\, f_{m+1}^j = \frac{q^{(j-n+3/2)/2}}{1-q} \sqrt{(1-q^n)(q^{n-2j-1}-1)}\, f_{m-1}^j$$

$$q^{J_3} f_m^j = q^m f_m^j(x) = q^{n-j} f_m^j$$

where $n = j+m$. Obviously, the operator J_3 is diagonal in the canonical basis. For the operators J_+ and J_- we have $J_+^* = J_-$.

3. The operator *I* and its spectrum

To study quantum and affine q-Krawtchouk polynomials, we require the operator I of the representation T_j, which has the form

$$I = \alpha q^{-3J_3/4} \left(\sqrt{pq^{J_3+j}-1}\, J_+ + J_- \sqrt{pq^{J_3+j}-1} \right) q^{-3J_3/4} - aq^{-2J_3} + bq^{-J_3} \tag{2}$$

where p is a fixed number such that $p > q^{-2j}$ and

$$\alpha = p^{-1} q^{-2j-1}(1-q) \qquad a = p^{-1} q^{-2j}(1+q^{-1}) \qquad b = p^{-1} q^{-j}(q^{-2j-1}+p+1).$$

In the canonical basis the operator I has the form of a Jacobi matrix:

$$\begin{aligned} I f_m^j = {} & p^{-1} q^{-2n-3/2} \sqrt{(1-q^{n+1})(q^{n-2j}-1)(pq^{n+1}-1)}\, f_{m+1}^j \\ & + p^{-1} q^{-2n+1/2} \sqrt{(1-q^n)(q^{n-2j-1}-1)(pq^n-1)}\, f_{m-1}^j \\ & - \left(p^{-1} q^{-2n}(1+q^{-1}) - p^{-1} q^{-n}(q^{-2j-1}+p+1)\right) f_m^j \end{aligned} \tag{3}$$

where, as before, $n = j+m$. It is clear that I is a well-defined symmetric operator.

Eigenfunctions $\psi_\lambda(x)$ of the operator I, $I\psi_\lambda(x) = \lambda\psi_\lambda(x)$, can be represented as linear combinations of the elements of the canonical basis:

$$\psi_\lambda(x) = \sum_{n=0}^{2j} p_n(\lambda) f_{n-j}^j. \tag{4}$$

By the action of the operator I upon both sides of this relation, one derives that

$$\begin{aligned} \sum_{n=0}^{2j} p_n(\lambda) \Big\{ & p^{-1} q^{-2n-3/2} \sqrt{(1-q^{n+1})(q^{n-2j}-1)(pq^{n+1}-1)}\, f_{n-j+1}^j \\ & + p^{-1} q^{-2n+1/2} \sqrt{(1-q^n)(q^{n-2j-1}-1)(pq^n-1)}\, f_{n-j-1}^j \Big\} \\ & - d_n f_m^j = \sum_{n=0}^{2j} p_n(\lambda) f_{n-j}^j \end{aligned}$$

where

$$d_n = p^{-1}q^{-2n}(1+q^{-1}) - p^{-1}q^{-n}(q^{-2j-1}+p+1).$$

Comparing coefficients of a fixed f_m^j, one obtains a three-term recurrence relation for the coefficients $p_n(\lambda)$:

$$p^{-1}q^{-2n-3/2}\sqrt{(1-q^{n+1})(q^{n-2j}-1)(pq^{n+1}-1)}\,p_{n+1}(\lambda) \\ - p^{-1}q^{-2n+1/2}\sqrt{(1-q^{n})(q^{n-2j-1}-1)(pq^{n}-1)}\,p_{n-1}(\lambda) - d_n p_n(\lambda) = \lambda p_n(\lambda).$$

We make here the substitution

$$p_n(\lambda) = \left(\frac{q^n(q^{-2j};q)_n}{(pq;q)_n(q;q)_n}\right)^{1/2} p'_n(\lambda)$$

and obtain the relation

$$q^{-2n-1}(1-q^{n-2j})p'_{n+1}(\lambda) + q^{-2n}(1-q^n)(1-pq^n)p'_{n-1}(\lambda) - d_n p'_n(\lambda) = p\lambda p'_n(\lambda)$$

which coincides with the recurrence relation for the quantum q-Krawtchouk polynomials

$$K_n^{\mathrm{qtm}}(\lambda; p, 2j; q) := {}_2\phi_1(q^{-n}, \lambda; q^{-2j}; q, pq^{n+1})$$

(see, for example, [15], section 3.14). Here ${}_2\phi_1$ is a basic hypergeometric function. Consequently, in (4) we have

$$p_n(\lambda) = \left(\frac{q^n(q^{-2j};q)_n}{(pq;q)_n(q;q)_n}\right)^{1/2} K_n^{\mathrm{qtm}}(\lambda; p, 2j; q). \tag{5}$$

Thus, in decomposition (4) the coefficients $p_n(\lambda)$ are given by (5) and we have

$$\psi_\lambda(x) = \sum_{n=0}^{2j}\left(\frac{q^n(q^{-2j};q)_n}{(pq;q)_n(q;q)_n}\right)^{1/2} K_n^{\mathrm{qtm}}(\lambda; p, 2j; q) f_{n-j}^j \\ = \sum_{n=0}^{2j}(-1)^{n/2}q^{(2j+3)n/4}\frac{(q^{-2j};q)_n}{(pq;q)_n^{1/2}(q;q)_n}K_n^{\mathrm{qtm}}(\lambda; p, 2j; q)x^n. \tag{6}$$

In order to find a spectrum of the operator I we could use the theory of operators, represented by Jacobi matrices (see, for example, [16], chapter VII), and find a spectrum of the operator I by using the orthogonality relation for quantum q-Krawtchouk polynomials. But we are going to show how the orthogonality relation itself for quantum q-Krawtchouk polynomials can be derived by using the operator I.

To search for a spectrum of the operator I, we first determine how the operator q^{J_3} acts upon eigenfunctions of the operator I. For this we use q-difference equation (3.14.5) in [15] for quantum q-Krawtchouk polynomials, represented in the form

$$q^n K_n^{\mathrm{qtm}}(q^{-y}) = p^{-1}q^y(q^{y-2j}-1)K_n^{\mathrm{qtm}}(q^{-y-1}) + (1-q^y)(1-p^{-1}q^{y-2j-1})K_n^{\mathrm{qtm}}(q^{-y+1}) \\ + p^{-1}[q^y(1-q^{y-2j}) + pq^y + q^{y-2j-1} - q^{2y-2j-1}]K_n^{\mathrm{qtm}}(q^{-y})$$

where $K_n^{\mathrm{qtm}}(q^{-y}) \equiv K_n^{\mathrm{qtm}}(q^{-y}; p, 2j; q)$. Next we multiply both sides of this relation by $b_n f_{n-j}^j$, where b_n is the coefficient of $K_n^{\mathrm{qtm}}(\lambda)$ in expression (5) for $p_n(\lambda)$, sum over n and take into account the decomposition (4). Since $q^{J_3} f_{n-j}^j = q^{-j+n} f_{n-j}^j$, we obtain the identity

$$q^{J_3+j}\psi_\lambda(x) = p^{-1}\lambda^{-1}(\lambda^{-1}q^{-2j}-1)\psi_{q^{-1}\lambda}(x) + (1-\lambda^{-1})(1-p^{-1}\lambda^{-1}q^{-2j-1})\psi_{q\lambda}(x) \\ + p^{-1}\lambda^{-1}(1-\lambda^{-1}q^{-2j} + p + q^{-2j-1} - \lambda^{-1}q^{-2j-1})\psi_\lambda(x). \tag{7}$$

In order to find values of λ, for which the functions $\psi_\lambda(x)$ are eigenfunctions of the operator I, we take into account the following. The operator I, acting upon eigenfunctions, leaves them invariant (up to a constant). Then by the action of the operator q^{J_3+j} upon eigenfunctions $\psi_\lambda(x)$ we must obtain $(2j+1)$-dimensional space or its subspaces. As we see from (7), when q^{J_3+j} acts on $\psi_\lambda(x)$, it maps it to a linear combination of $\psi_{q\lambda}(x)$, $\psi_{q^{-1}\lambda}(x)$ and $\psi_\lambda(x)$. By the action on these functions again, we obtain additional functions $\psi_{q^2\lambda}(x)$ and $\psi_{q^{-2}\lambda}(x)$. Further by the action of q^{J_3+j}, we again obtain new functions. This procedure cannot be continued infinitely. Namely, some coefficients in (7) must vanish when we increase (decrease) the value of λ. We see from (7) that a coefficient in (7) vanishes under increasing and under decreasing of the value of λ only when $\lambda = 1$ and $\lambda = q^{-2j}$.

Suppose that $\psi_\lambda(x)$ with $\lambda = 1$ is an eigenfunction of the operator I. Putting $\lambda = 1$ into (7), we see that the operator q^{J_3+j} maps this eigenfunction to a linear combination of $\psi_1(x)$ and $\psi_{q^{-1}}(x)$. Thus, $\psi_{q^{-1}}(x)$ also belongs to the representation space $\mathcal{H}_j$. The action of the operator q^{J_3+j} upon the function $\psi_{q^{-1}}(x)$ leads to the linear combination of the functions $\psi_1(x)$, $\psi_{q^{-1}}(x)$ and $\psi_{q^{-2}}(x)$. Thus, $\psi_{q^{-2}}(x)$ also belongs to the representation space. Continuing this procedure further, we conclude that the $(2j+1)$ functions $\psi_{q^{-k}}(x)$, $k = 0, 1, 2, \ldots, 2j$, belong to the representation space. Note that the action of q^{J_3+j} upon $\psi_{q^{-2j}}(x)$ does not lead to new elements of the representation space. Thus, under the condition that the function $\psi_{q^0}(x)$ belongs to the representation space, we have obtained $(2j+1)$ linear independent elements of the space $\mathcal{H}_j$. If we start from $\psi_{q^{-2j}}(x)$, we will obtain the same functions.

On the other hand, it is easy to check that if we take the function $\psi_\lambda(x)$ with $\lambda \neq q^{-n}$, $n = 0, 1, 2, \ldots, 2j$, then by the action of the operator q^{J_3+j}, we will obtain infinite-dimensional space because in this case the coefficients in formula (7) do not vanish. Thus, only the functions $\psi_{q^{-n}}(x)$, $n = 0, 1, 2, \ldots, 2j$, belong to the representation space and constitute a basis in this space.

Proposition 1. *The spectrum of the operator I coincides with the set of points q^{-n}, $n = 0, 1, 2, \ldots, 2j$. The corresponding eigenfunctions are given by formula (6).*

4. Orthogonality relation for quantum q-Krawtchouk polynomials

Now one can derive the orthogonality relation for the quantum q-Krawtchouk polynomials by employing the same method as in [10]. For this we use the operator q^{J_3}. Introducing the notation $e_n(x) \equiv \psi_{q^{-n}}(x)$, $n = 0, 1, 2, \ldots, 2j$, we find that

$$\begin{aligned} q^{J_3+j} e_n = {} & -p^{-1}q^n(1-q^{n-2j})e_{n+1} + (1-q^n)(1-p^{-1}q^{-2j+n-1})e_{n-1} \\ & + p^{-1}q^n(1-q^{-2j+n}+p+q^{-2j-1}-q^{-2j+n-1})e_n. \end{aligned}$$

As we see, the matrix of the operator q^{J_3+j} in the basis e_n, $n = 0, 1, 2, \ldots, 2j$, is not symmetric (as we know, in the canonical basis this matrix is diagonal and, therefore, symmetric). The reason for this is that the matrix of the transition from the canonical basis $\{f_m^j\}$ to the basis of eigenfunctions $\{e_n\}$ is not unitary. It is equivalent to the statement that the basis $e_n = \psi_{q^{-n}}(x)$, $n = 0, 1, 2, \ldots, 2j$, is not normalized. Let us normalize it. For this we have to multiply e_n by the corresponding numbers c_n. Let $\hat{e}_n = c_n e_n$, $n = 0, 1, 2, \ldots, 2j$, be a normalized basis. Then the matrix of the operator q^{J_3+j} is symmetric in this basis. Since in the basis $\{\hat{e}_n\}$ the operator q^{J_3+j} has the form

$$\begin{aligned} q^{J_3+j}\hat{e}_n = {} & -c_{n+1}^{-1}c_n p^{-1}q^n(1-q^{n-2j})\hat{e}_{n+1} + c_{n-1}^{-1}c_n(1-q^n)(1-p^{-1}q^{-2j+n-1})\hat{e}_{n-1} \\ & + p^{-1}q^n(1-q^{-2j+n}+p+q^{-2j-1}-q^{-2j+n-1})\hat{e}_n \end{aligned}$$

its symmetricity means that

$$c_{n+1}^{-1}p^{-1}q^n c_n(1-q^{n-2j}) = -c_n^{-1}c_{n+1}(1-p^{-1}q^{n-2j})(1-q^{n+1})$$

that is,

$$\frac{c_n}{c_{n-1}} = \sqrt{\frac{-(1-q^{n-2j-1})p^{-1}q^{n-1}}{(1-p^{-1}q^{n-2j-1})(1-q^n)}}.$$

Taking into account relation (5), we derive from here that

$$c_n = c\left(\frac{(-p)^{-n}q^{n(n-1)/2}(q^{-2j};q)_n}{(q;q)_n(p^{-1}q^{-2j};q)_n}\right)^{1/2} = c\left(\frac{(-1)^n q^{n(n-1)/2}(q;q)_N(pq;q)_{N-n}}{(q;q)_n(q;q)_{N-n}(pq;q)_N}\right)^{1/2} \tag{8}$$

where c is a constant.

Thus, the relation

$$\hat{e}_n(x) = \sum_{m=0}^{2j} c_n p_m(q^{-n}) f^j_{m-j}$$

where $p_m(q^{-n})$ are given by formula (5), connects two orthonormal bases in the representation space $\mathcal{H}_j$. This means that the matrix (a_{mn}), $m, n = 0, 1, 2, \ldots, 2j$, with entries

$$a_{mn} = c\left(\frac{(-p)^{-n}q^{n(n-1)/2}(q^{-2j};q)_n}{(q;q)_n(p^{-1}q^{-2j};q)_n}\frac{q^m(q^{-2j};q)_m}{(pq;q)_m(q;q)_m}\right)^{1/2} K_m^{\mathrm{qtm}}(q^{-n};p,2j;q)$$

is unitary under appropriate choice of the constant c. In order to calculate this constant, we use the relation $\sum_{n=0}^{2j}|a_{mn}|^2 = 1$ at $m = 0$. Denoting this sum by A, we have

$$A = c^2\sum_{n=0}^{2j}\frac{(q^{-2j};q)_n}{(p^{-1}q^{-2j};q)_n(q;q)_n}(-1)^n q^{n(n-1)/2}p^{-n} = c^2{}_1\phi_1(q^{-2j};p^{-1}q^{-2j};q,p^{-1}).$$

Using relation (II.5) of Appendix II in [13], one reduces it to

$$A = c^2\frac{(p^{-1};q)_\infty}{(p^{-1}q^{-2j};q)_\infty} = \frac{(-p)^N q^{N(N+1)/2}}{(pq;q)_N}.$$

where $N = 2j$. Thus, one has

$$c^2 = (pq;q)_N/(-p)^N q^{N(N+1)/2} \tag{9}$$

and the relation $\sum_{n=0}^{2j} a_{mn}a_{m'n} = \delta_{mm'}$ leads to the following orthogonality relation for the quantum q-Krawtchouk polynomials $K_n^{\mathrm{qtm}}(q^{-m}) \equiv K_n^{\mathrm{qtm}}(q^{-m};p,N;q)$, $N = 2j$:

$$\sum_{n=0}^{N}\frac{(-p)^{-n}q^{n(n-1)/2}(q^{-2j};q)_n}{(q;q)_n(p^{-1}q^{-2j};q)_n}K_m^{\mathrm{qtm}}(q^{-n})K_{m'}^{\mathrm{qtm}}(q^{-n}) = h_n\delta_{mm'} \tag{10}$$

where

$$h_n = \frac{(-p)^N q^{N(N+1)/2}}{(pq;q_N)}\frac{(pq;q)_m(q;q)_m}{q^m(q^{-2j};q)_m}.$$

After some transformations, using relations of Appendix I in [13], it reduces to the known one, derived analytically (see, for example, [13], chapter 7).

5. Dual quantum q-Krawtchouk polynomials

The relation $\sum_{m=0}^{2j} a_{mn}a_{mn'} = \delta_{nn'}$ determines an orthogonality relation for the polynomials, dual to the quantum q-Krawtchouk polynomials. We denote them by $k_n(q^{-m};p,N;q)$. They are given by the formula

$$k_n(q^{-x};p,N;q) = {}_2\phi_1(q^{-x},q^{-n};q^{-N};q,pq^{x+1}).$$

The orthogonality relation for them has the form

$$\sum_{m=0}^{N} \frac{q^m (q^{-2j}; q)_m}{(pq; q)_m (q; q)_m} k_n(q^{-m}) k_{n'}(q^{-m}) = \frac{(-p)^N q^{N(N+1)/2} (q; q)_n (p^{-1} q^{-2j}; q)_n}{(pq; q)_N (-p)^{-n} q^{n(n-1)/2} (q^{-2j}; q)_n} \delta_{nn'}. \tag{11}$$

Note that the polynomials $k_n(q^{-x}; p, N; q)$ are multiple of the affine q-Krawtchouk polynomials $K_n^{\rm aff}(q^{-x}; p, N; q)$, which are given by the formula

$$\begin{aligned} K_n^{\rm aff}(q^{-x}; p', N; q) &:= {}_3\phi_2(q^{-n}, 0, q^{-x}; p'q, q^{-N}; q, q) \\ &= \frac{(-p'q)^n q^{n(n-1)/2}}{(p'q; q)_n} {}_2\phi_1(q^{-n}, q^{x-N}; q^{-N}; q, q^{-x}/p'). \end{aligned} \tag{12}$$

Namely, we have

$$k_n(q^{-x}; p, N; q) = \frac{(p'q; q)_n}{(-p'q)^n q^{n(n-1)/2}} K_n^{\rm aff}(q^{N-x}; p', N; q) \qquad p' = q^{-N-1}/p.$$

Substituting this expression for $k_n(q^{-x}; p, N; q)$ into the orthogonality relation (11), one derives the orthogonality relation for affine q-Krawtchouk polynomials $K_n^{\rm aff}(q^{-x}; p', N; q)$,

$$\sum_{m=0}^{N} \frac{(p'q; q)_m (q; q)_N}{(q; q)_m (q; q)_{N-m}} (p'q)^{-m} K_n^{\rm aff}(q^{-m}) K_{n'}^{\rm aff}(q^{-m}) = \frac{(q; q)_n (q; q)_{N-n}}{(p'q; q)_n (q; q)_N} (p'q)^{n-N} \delta_{nn'}$$

which coincides with the known one (see, for example, [13], chapter 7).

Note that from the first expression in formula (12) for the affine q-Krawtchouk polynomials $K_n^{\rm aff}(q^{-x}; p', N; q)$ it follows that these polynomials are self-dual. But as we have seen the quantum q-Krawtchouk polynomials $K_n^{\rm qtm}(q^{-x}; p, N; q)$ have duals, which are multiple of affine q-Krawtchouk polynomials.

6. Realizations of T_j related to quantum q-Krawtchouk polynomials

The representation T_j is realized on the finite-dimensional Hilbert space $\mathcal{H}_j$ of polynomials in x. Let us construct another realization of this representation, related to the quantum q-Krawtchouk polynomials.

We introduce a finite-dimensional Hilbert space $\mathfrak{l}_p^2$, which consists of finite sequences $\mathbf{a} = \{a_k | k = 0, 1, 2, \ldots, 2j\}$. The scalar product in this Hilbert space is naturally defined as

$$\langle \mathbf{a}, \mathbf{a}' \rangle_0 = c^2 \sum_{n=0}^{2j} \frac{(-p)^{-n} q^{n(n-1)/2} (q^{-2j}; q)_n}{(q; q)_n (p^{-1} q^{-2j}; q)_n} a_n \overline{a'_n}$$

where c is given by formula (9) and the weight function coincides with the orthogonality measure in (10). Then the sequences of values of the polynomials

$$p_n(\lambda) = \left(\frac{q^n (q^{-2j}; q)_n}{(pq; q)_n (q; q)_n} \right)^{1/2} K_n^{\rm qtm}(\lambda; p, 2j; q) \tag{13}$$

from (5) on the set $\{q^{-k} | k = 0, 1, 2, \ldots, 2j\}$ form an orthonormal basis in $\mathfrak{l}_p^2$. We denote these sequences by $\{p_n(q^{-k})\}$, $n = 0, 1, 2, \ldots, 2j$.

Let $\mathcal{H}_j$ be the Hilbert space as mentioned in section 2 and $f(x) = \sum_{n=0}^{2j} a_n f_{n-j}^j(x)$ be an expansion of $f \in \mathcal{H}_j$ with respect to the orthonormal basis (1). With every function $f \in \mathcal{H}_j$ we associate the sequence $\{F(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}$ such that

$$F(q^{-k}) = \langle f(x), \psi_{q^{-k}}(x) \rangle$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathcal{H}_j$ and $\psi_{q^{-k}}(x)$, $k = 0, 1, 2, \ldots, 2j$, are the eigenfunctions of the operator I. This defines a linear mapping $\Phi : f(x) \to \{F(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}$ from $\mathcal{H}_j$ to the Hilbert space $\mathfrak{l}_p^2$. The following proposition is easily proved.

Proposition 2. *The mapping $\Phi : f(x) \to \{F(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}$ establishes an invertible isometry between the Hilbert spaces $\mathcal{H}_j$ and $\mathfrak{l}_p^2$.*

It is directly checked that the isometry Φ maps basis elements f^j_{n-j} in the space $\mathcal{H}_j$ to the basis elements $\{p_n(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}$ in the space $\mathfrak{l}_p^2$.

For the action of the operator I on the elements $\{F(q^{-k})\}$ of the space $\mathfrak{l}_p^2$, we have

$$IF(q^{-k}) = \langle If(x), \psi_{q^{-k}}(x)\rangle = \langle f(x), I\psi_{q^{-k}}(x)\rangle = q^{-k}\langle f(x), \psi_{q^{-k}}(x)\rangle = q^{-k}F(q^{-k})$$

that is,

$$I\{p_n(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\} = \{q^{-k}p_n(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}. \tag{14}$$

Taking into account formulae (13) and (14), as well as the recurrence relation for quantum q-Krawtchouk polynomials, we deduce that

$$\begin{aligned} I\{p_n(q^{-k})\} &= p^{-1}q^{-2n-3/2}\sqrt{(1-q^{n+1})(q^{n-2j}-1)(pq^{n+1}-1)}\{p_{n+1}(q^{-k})\} \\ &\quad + p^{-1}q^{-2n+1/2}\sqrt{(1-q^n)(q^{n-2j-1}-1)(pq^n-1)}\{p_{n-1}(q^{-k}\} \\ &\quad - (p^{-1}q^{-2n}(1+q^{-1}) - p^{-1}q^{-n}(q^{-2j-1}+p+1))\{p_n(q^{-k})\}. \end{aligned} \tag{15}$$

Comparing this formula with formula (3), we see that the operator I acts upon the basis elements $\{p_n(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}$ by the same formula as upon the basis functions $f^j_{n-j}(x)$ of the space $\mathcal{H}_j$. We also have

$$q^{J_3}\{p_n(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\} = q^{n-j}\{p_n(q^{-k}) \,|\, k = 0, 1, 2, \ldots, 2j\}. \tag{16}$$

The operators (15) and (16) determine uniquely all other operators of the representation T_j on $\mathfrak{l}_p$. In particular, we have

$$J_+\{p_n(q^{-k}) \,|\, k = 0, 1, \ldots, 2j\} = \sqrt{[2j-n]_q[n+1]_q}\{p_{n+1}(q^{-k}) \,|\, k = 0, 1, \cdots, 2j\}$$
$$J_-\{p_n(q^{-k}) \,|\, k = 0, 1, \ldots, 2j\} = \sqrt{[2j-n+1]_q[n]_q}\{p_{n-1}(q^{-k}) \,|\, k = 0, 1, \cdots, 2j\}.$$

The results of this section allow us to prove the following assertion.

Proposition 3. *Let $p_n(\lambda)$ be the polynomials determined in (13). Then in the Hilbert space $\mathcal{H}_l$ we have*

$$p_n(I)f^j_{-j} = f^j_{-j+n}. \tag{17}$$

Proof. The isometry $\Phi : \mathcal{H}_j \to \mathfrak{l}_p^2$ maps $f^j_{-j} \equiv 1$ to $p_0(\lambda) \equiv 1$. By formula (14) we have $Ip_0 \equiv I_1 1 = q^{-k}$. Therefore, $p_n(I)p_0 = p_n(q^{-k})$. Applying the mapping Φ^{-1} to this identity, one obtains the desired relation (17). Hence the proposition is proved. □

Note that f^j_{-j} in (17) is the lowest canonical vector. Thus, by the action of the polynomials $p_n(I)$, $n = 0, 1, 2, \ldots, 2j$, upon this vector, we obtain all weight vectors of the representation space.

7. Biorthogonal system of functions

From the very beginning one could consider an operator

$$I_1 = \alpha q^{-3J_3/4}(J_+ + J_-(pq^{J_3+j} - 1))q^{-3J_3/4} - aq^{-2J_3} + bq^{-J_3}$$

instead of the operator I, where, as before, p is a fixed number such that $p > q^{-2j}$ and

$$\alpha = p^{-1}q^{-2j-1}(1-q) \qquad a = p^{-1}q^{-2j}(1+q^{-1}) \qquad b = p^{-1}q^{-j}(q^{-2j-1} + p + 1).$$

This operator is well defined, but it is not self-adjoint. Repeating the reasoning of section 3, we find that eigenfunctions of I_1 are of the form

$$\chi_{q^{-m}}(x) = \sum_{n=0}^{2j} \mathrm{i}^n q^{(2j+3)n/4} \frac{(q^{-2j};q)_n}{(pq;q)_n(q;q)_n} K_n^{\mathrm{qtm}}(q^{-m}; p, 2j; q)x^n. \tag{18}$$

The right-hand side can be summed with the help of formula (3.14.12) in [15]. We thus have

$$\chi_{q^{-m}}(x) = (\mathrm{i}q^{-m+(2j+3)/4}x; q)_m \cdot {}_2\phi_1(q^{m-N}, 0; pq; q, \mathrm{i}q^{-m+(2j+3)/4}x).$$

Now we consider another operator

$$I_2 := \alpha q^{-3J_3/4}((pq^{J_3+j} - 1)J_+ + J_-)q^{-3J_3/4} - aq^{-2J_3} + bq^{-J_3}$$

where α, a and b are the same as above. This operator is adjoint to the operator $I_1 : I_2^* = I_1$. Repeating the reasoning of section 3, we find that eigenfunctions of I_2 have the form

$$\varphi_{q^{-m}}(x) = \sum_{n=0}^{2j} \mathrm{i}^n q^{(2j+3)n/4} \frac{(q^{-2j};q)_n}{(q;q)_n} K_n^{\mathrm{qtm}}(q^{-m}; p, 2j; q)x^n. \tag{19}$$

The right-hand side of (19) can be summed with the help of formula (3.14.11) in [15]. We thus have

$$\varphi_{q^{-m}}(x) = (\mathrm{i}q^r x; q)_{N-m} \cdot {}_2\phi_1(q^{-m}, pq^{N-m+1}; 0; q, \mathrm{i}q^r x)$$

where $r = m - N + (2j+3)/4$.

Let us denote by $\Psi_m(x)$, $m = 0, \pm 1, \pm 2, \ldots, 2j$, the functions

$$\Psi_m(x) = c_m \chi_{q^{-m}}(x) \qquad m = 0, 1, 2, \ldots, 2j \tag{20}$$

and by $\Phi_m(x)$, $m = 0, \pm 1, \pm 2, \ldots, 2j$, the functions

$$\Phi_m(x) = c_m \varphi_{q^{-m}}(x) \qquad m = 0, 1, 2, \ldots, 2j \tag{21}$$

where c_m are given by formula (8).

Writing down the decompositions (18) and (19) for the functions $\Psi_m(x)$ and $\Phi_m(x)$ in terms of the orthonormal basis f_{n-j}^l, $n = 0, 1, 2, \ldots, 2j$, of the Hilbert space $\mathcal{H}_j$ and taking into account the orthogonality relations (10), one finds that

$$\langle \Psi_m(x), \Phi_n(x) \rangle = \delta_{mn} \qquad m, n = 0, 1, 2, \ldots, 2j.$$

Therefore, one can formulate the following statement.

Theorem. *The set of functions $\Psi_m(x)$, $m = 0, 1, 2, \ldots, 2j$, and the set of functions $\Phi_m(x)$, $m = 0, 1, 2, \ldots, 2j$, form sets of functions biorthogonal with respect to the scalar product in the Hilbert space $\mathcal{H}_j$.*

8. Concluding remarks

In this paper we have studied the operator I, represented in the canonical basis by a Jacobi matrix. Its diagonalization is effected by employing quantum q-Kravchuk polynomials. We have explicitly found a spectrum of this operator. As an immediate physical application of this result one may try to construct a finite model of the quantum harmonic oscillator [17, 18] in the following way. With the help of the operator I one constructs another operator $I' := \mathrm{i}[J_3, I]$. Then it turns out that $[J_3, I'] = \mathrm{i}I$. Consequently, in the irreducible representation T_j of the quantum algebra $U_q(\mathrm{su}_2)$ one may use the operators $H := J_3 + j + 1/2$ (with the spectrum $n + 1/2, n = 0, 1, 2, \ldots, 2j$), $Q = I$ and $P = I'$. This means that these three operators satisfy the appropriate quantum-mechanical commutation relations

$$[Q, H] = \mathrm{i}P \qquad [H, P] = \mathrm{i}Q$$

with finite spectra for the position and momentum operators Q and P. A study of this model is in progress.

Another possibility for constructing a finite quantum-mechanical system is to employ the operator q^{J_3} and find two more operators, which commute with q^{J_3} in an appropriate way. However, this model is more complicated than the previous one, and it will be addressed in a future publication.

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